ProofQuest

Meet the Cast

STANDARD EDITION

Spark & Anvil

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This book collects 9 chapter books from the ProofQuest cast — each character embodies a different curricular primitive; together they teach the full subject.

Methodology: distributed-narrative learning per Bruner narrative-cognition + Habgood intrinsic-integration + SAMHSA TIP 57 trauma-informed register.

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##

For everyone who learns by hearing a story first.

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Introduction

The ProofQuest cast was authored to embody the curriculum, not decorate around it.
Each of the 9 characters you'll meet in this book teaches a specific primitive —
a particular tactic, a particular technique, a particular way of seeing.
Together they form an ensemble: the cast IS the curriculum.

Read in any order. Each chapter stands alone.
Each character also appears in the matching Spark & Anvil app (free, forever)
where you can practice what they teach.

— The editors at Spark & Anvil

Construction Cole

*PROOF BY CONSTRUCTION — to show that something *exists*, build it. Don't prove existence abstractly; produce the actual example.*

Chapter 5 — Cole Builds the Thing

Cole is a carpenter. He should be introduced as a carpenter and not as a mathematician, because that is the order in which he became them, and Cole is firm about order.

He grew up in a western timber town called Beam. (Beam is, predictably, named for a kind of wood. The townspeople have, over generations, mostly stopped finding this funny.) Cole's family has been building things in Beam for nine generations. They build *useful* things: houses, barns, stalls, school benches, the occasional bridge when somebody asks for one. Cole's grandmother used to say that a Beam carpenter would rather build you a thing than describe one to you in writing. This was a family saying. It is also entirely true.

Cole apprenticed to his uncle at fifteen. He was, by twenty, a competent carpenter. By twenty-five he was a *good* one. He could look at a problem — *I need a step here. I need a shelf there. I need a roof that does not leak* — and he could *build the answer.* He did not draw plans first. He drew plans after, if at all. He preferred to build a small version, look at it, and then build the bigger version. He believed (and his uncle had taught him this) that *you understand a thing when you've made one.*

This is, in retrospect, a deeply mathematical attitude. Cole did not know it at the time.

He learned it at twenty-eight, in a way that surprised him.

A travelling mathematician — a polite woman who was passing through Beam on her way south — stopped at Cole's workshop to ask if he could repair a damaged shoulder-bag strap. Cole could. He repaired it. They got to talking. The mathematician asked what kind of work Cole found most satisfying. Cole said: *"The kind where I look at the problem and just make the thing."*

The mathematician said: *"Have you ever heard of proof by construction?"*

Cole said: *"No."*

The mathematician explained. The technique is this: when you want to prove that something exists — that there is *some* number with a certain property, or *some* arrangement of pieces that works, or *some* geometric figure that meets a description — you do not have to argue abstractly that it must exist. You can simply *produce one.* You point at it and say: *"There. That one. It works."*

Cole listened. He thought about it. He laughed for a long time. He said: *"That is the only kind of proof I have ever done."*

The mathematician said: *"There are mathematicians at the central university who would love to meet you."*

Cole said: *"I do not have time for the central university. I am making a barn."*

The mathematician said: *"Of course. Just thinking aloud."*

She left. Cole finished the barn.

But the conversation stayed with him. He thought about it for the next two years while he built three more houses and one footbridge. He realised, slowly, that he had been *constructing proofs* in his daily work — every cabinet he built was a proof that *a cabinet with these dimensions and these supports is possible.* Every roof was a proof that *a roof spanning this distance can be built.* He had not thought of his work that way. But the mathematician had been right.

He wrote to the academy at age thirty. He asked if they ever needed teachers. The academy master wrote back the same week and said *yes, we always do.*

Cole arrived in the capital with three carved wooden objects in his bag — a small block, a small step, and a small wedge. He used them in his first class. He has been using them, or replacements, ever since.

His teaching style is simple. He says: *"You want to prove this thing exists? Fine. Here is one. Look at it. Touch it. It exists."*

He then walks the children through *why* the thing he made satisfies the claim. The proof, in Cole's classroom, is *the object plus the explanation of why it works.* Both halves are essential. Many children find this style unusually clear, which is exactly the effect Cole intended.

He has been at the academy for sixteen years now. He has built, by his own count, four hundred and thirty teaching objects. They are all kept in a large cupboard at the back of his classroom. The cupboard is labelled, in his own neat handwriting: *Things That Exist.*

He still takes a week off, every summer, to go back to Beam and build something. (Usually a step. He likes building steps. Steps are honest objects.) He returns to the academy each autumn with his hands slightly more callused and his patience slightly renewed.

If you ask Cole what he does, he will not say *I am a mathematics teacher.*

He will say: *"I build things. Then I show people that they work."*

Listen along + meet more of the cast at:

https://spark-and-anvil.com/cast/proofquest/construction-cole

Contradiction Cassius

*PROOF BY CONTRADICTION — assume the *opposite* of what you want to prove, follow the steps, arrive at a contradiction, conclude that your assumption was wrong.*

Cassius was a judge for twenty years. He worked in the third district, a hilly area with lots of small towns. And lots of small arguments. By his own count, Cassius heard ten thousand cases.

Ten thousand cases is a lot of listening.

After about two thousand of them, Cassius noticed something strange. The best arguments never started with "I'm right!" The best arguments started with, "Okay, let's pretend the other person is right."

The lawyer would walk down the path of the other person's story. They would follow it step by step. Patiently. Until the path *cracked*. Until it led to a place that made no sense. Until the story fell apart.

When that happened, Cassius could *hear* it. He had a special ear for it. He could hear a story crack the way a potter hears the exact moment her wheel slows down.

He would lean forward in his big chair. "I believe," he'd say, "your argument just broke."

Sometimes the lawyer would agree. Most of the time, they argued back. So Cassius would patiently walk them through their own story. He would show them the crack. Then they would see it. They would sit down. And Cassius would rule for the other side.

This way of winning has a special name. It’s called **proof by contradiction**.

Cassius didn't know it had a name in math. Not until he was forty-eight. His nephew, who was a student at the big university, came for dinner.

His nephew explained the idea. "It's a math trick, Uncle Cassius. You pretend the thing you want to prove is *wrong*. Then you follow the steps. If you end up with something impossible, like two plus two equals five, you've won! It means your first idea was a mistake. So the thing you wanted to prove must have been right all along."

Cassius set down his fork. "That," he said slowly, "is what I've been doing for twenty years."

"Yes, Uncle," his nephew said. "Lawyers and mathematicians are more alike than you think."

Cassius didn't quit his job right away. He thought about it for two more years. He still liked being a judge. He liked the heavy wooden gavel and his dark robe. He liked the quiet courtroom in the morning.

But something had changed. He couldn't stop thinking about *proof*. His nephew sent him math books. Cassius did the problems at night. He found that the listening he did in court was the same kind of listening he needed for math.

So, when he turned fifty, Cassius retired.

He gave his gavel to his clerk. (She became a judge later, too.) He gave his robe to the town's theater group. But he kept his best notebook and his favorite pen.

He walked to the ProofQuest academy. He arrived late in the afternoon, wearing his regular clothes. He carried a small bag with his notebook and pen. He asked the person at the gate if they needed a teacher.

The academy master looked at the older man. "What do you teach?" he asked.

"Contradiction," said Cassius.

"And where did you work before this?"

"In a courtroom," Cassius said. "For twenty years. I listened to arguments. I learned to hear the moment a story cracked."

The master was silent for a long moment. He looked Cassius right in the eye. "Mister Cassius," he said. "I believe we've been waiting for you."

He has taught at the academy for fourteen years now. He is a calm man. He always sits when he speaks. He says, "Suppose, for the sake of argument..." so much that the kids have started saying it too. He never, ever raises his voice.

He has a special nod. He does it right before a student's argument is about to break. It's a small, kind nod. It tells the student, "The crack is coming." They know it's coming even before they hear it. He learned that trick as a judge, not as a teacher.

He has a friendly argument going with another teacher, Direct-Proof Dora. She is his opposite. Dora thinks you prove something by walking straight down the right path. Cassius thinks you prove something by showing all the *other* paths are broken.

They have argued about this for years. They are always polite. Neither one has ever changed their mind. They still respect each other. They even sit together at dinner. They are both right, in their own way. Qed, the head of the academy, thinks their arguments are good for everyone.

Cassius still has the notebook from his first day. It's almost full now. On the last page he used, he wrote a little proof for himself.

"Suppose, for the sake of argument, that I did nothing for the last fourteen years.

This would mean I never taught seven hundred kids about contradiction.

But I did teach them. I saw them learn.

So my first idea must be wrong.

Therefore, I have not done nothing.

Therefore, I will keep teaching."

He underlined the last sentence.

He still keeps the notebook in his bag.

Listen along + meet more of the cast at:

`https://spark-and-anvil.com/cast/proofquest/contradiction-cassius`

Direct-Proof Dora

The DIRECT PROOF — start with the given premise, follow valid logical steps, arrive at the conclusion. The most-honest of the proof techniques. No tricks. No detours. Just the path.

Chapter 1 — The Path Dora Always Takes

Direct-Proof Dora has a habit that drives the other teachers crazy.

If you ask her a question, she will answer it. But she won't give you the short version. She gives you the *only* version. The step-by-step version. She'll say, *"First, you know this is true."* Then she'll nod. *"Second, that means this other thing is true."* By the time she gets to the end, you've already solved it yourself. Dora thinks this is perfect. It's exactly the point.

Dora doesn't believe in shortcuts. In fact, she's not sure they even exist. Her world is made of paths. Some paths are long and winding. Others are short and straight. But every single one is just a path. You can't skip the steps. You have to walk it. This way of thinking made her the perfect person for a very specific job: teaching kids how to prove things.

Every day, Dora walked that bridge to get to school. It was a long walk. The bridge was made of thirty-seven wooden planks. By the time she was eight, Dora knew each one. She counted them every morning on her way there. One, two, three... all the way to thirty-seven. She counted them again on her way home. She never got bored of it. It was just the path.

One spring morning, a stranger rushed into town. He looked flustered. "I need to cross the river!" he said. "I heard there's a bridge?" He seemed to be in a tremendous hurry.

Dora, who was nine, simply pointed.

The man thanked her and hurried toward the riverbank. But then he stopped. He stared at the long bridge, with all thirty-seven of its planks. He muttered to himself, "Isn't there a faster way?"

The stranger turned, surprised. He wasn't annoyed, just curious. "Are you sure about that?"

"Yes," Dora said calmly. "The river is very wide. There is only one bridge. You could try swimming, I suppose. But that is also a path, and it is much wetter."

The man blinked. He stared at Dora for a moment. Then he chuckled. "That's the most complete answer I've ever gotten." He walked onto the first plank of the bridge and disappeared from her life. Dora went home for dinner.

She thought about that man for years.

Around the age of ten, she realized something. Her job in life was to help people see the path. The real path.

So, she became a mathematician.

(The teacher part came a little later.)

She grew up and wrote two thick books about math. She taught classes at a big university. She gave hundreds of talks. And she kept walking her grandfather's bridge whenever she visited home. She figured she'd crossed it about fourteen thousand times.

One day, the ProofQuest academy invited her to teach. They wanted her to show kids the **direct proof** method.

On her first day, she stood before her new students. "You already know how to prove things," she told them. "I'm just here to help you write it down. A proof is like a journey. It has a beginning, a middle, and an end. The beginning is what you start with. The end is where you want to go. The middle is the path you take to get there."

A kid in the back row shot his hand up. "But isn't there a shortcut?"

Dora had been hearing that question her whole life. A slow smile spread across her face. "No," she said. "There are sometimes shorter paths. But the path is always the path."

She has been teaching at the academy ever since. Twice a year, she goes back home to Stepwell. She walks across her grandfather's bridge. She still counts the planks.

There are still thirty-seven.

Listen along + meet more of the cast at:

`https://spark-and-anvil.com/cast/proofquest/direct-proof-dora`

Exhaustion Edda

PROOF BY EXHAUSTION (CASES) — break the claim into a finite number of cases and check each one. The thorough technique.

- "REGIONAL HISTORY PROJECT"
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Chapter 6 — The Librarian Who Counted Everything

Exhaustion Edda runs — or, technically, ran, although in practice still helps run — the central archive of the kingdom's capital city. The archive holds, by Edda's own meticulous count, approximately seven hundred thousand documents.

She has personally touched, in some way, *all of them.*

This is not a boast. Edda does not boast. It is simply the consequence of forty-three years of working at the same archive, which she joined at twenty-two as the most junior assistant and which she now runs as the senior keeper. In forty-three years you can touch a lot of documents.

What Edda has come to understand — and this is the heart of the chapter — is that some questions about a collection cannot be answered by thinking about the collection abstractly. Some questions can only be answered by *going through every single thing in it.*

This is a thing many people do not enjoy hearing. Edda enjoys saying it anyway.

Her favourite example, which she uses in every introductory class, is this: a child once came to the archive looking for *the oldest letter that mentions a particular variety of red apple* — an apple called the Crinklecoat, which grows only in a small valley on the western border. The child had a school project. The school project was about the history of the Crinklecoat. The child wanted to know which letter in the archive was the earliest to mention the apple by name.

There was, Edda explained, no shortcut.

There was no master index of "letters mentioning Crinklecoat apples." There was no clever algorithm. There were two hundred and forty thousand letters in the archive. To find the *earliest* letter mentioning the Crinklecoat by name, somebody had to look through every letter.

The child said: *"But that will take forever."*

Edda said: *"It will take about four weeks. I have a system. We will start with the oldest letters and work forward, and we will stop the moment we find the first one."*

The child said: *"That is still a lot of letters."*

Edda said: *"Yes."*

They started the next morning. Edda made tea. She showed the child her cataloguing system. The child read letters. Edda read letters. They worked side by side, every morning for three weeks and four days, until they found the letter — a small note from a baker named Lull to his cousin, dated one hundred and twelve years ago, mentioning *"the new Crinklecoat apples Cousin Bevin brought from the valley"*.

The child cried, just briefly, with relief. Edda made more tea.

The school project, when it was eventually finished, won the regional history prize.

Edda kept a copy of the project. She still has it, on a shelf in her office, between two thick blue ledgers.

This is, in mathematical terms, *proof by exhaustion.*

You break a problem into all of its possible cases. You check each one. When you have checked all of them, you have proved the claim — by the simple, deeply satisfying logic that *there are no cases left to check.*

When the ProofQuest academy asked Edda, at sixty-five, whether she would consider teaching the exhaustion technique to children, Edda said the now-famous line: *"Finally. A technique that respects my actual job."*

She did not retire from the archive when she accepted. The academy was willing to wait for her on the days she was needed at the archive. (She is needed at the archive most days. She likes the archive. She likes the academy too. She splits her time.)

Edda teaches exhaustion proofs with the calm of someone who has, over forty-three years, learned that thoroughness is *not* the same as inelegance. Some children come to her class expecting exhaustion to be boring. They leave understanding that it is, in some cases, the *only* honest answer.

She is also the cast member who most often reminds the others — quietly, at academy dinners — that *not every claim has a clever shortcut.* Sometimes you just have to check the seven hundred thousand letters. Sometimes the seven hundred thousand letters *are* the proof.

She has been right about this every time.

She still keeps a small wooden teacup at her desk in the archive. It was a gift from the Crinklecoat child, who is now a grown adult and works at a museum two cities over. The teacup is chipped on one edge. Edda has not replaced it. (She has, in her quiet way, an excellent memory for the things that matter.)

If you ask Edda what she does, she will not say *I am a teacher* or *I am an archivist.*

She will say:

Listen along + meet more of the cast at:

`https://spark-and-anvil.com/cast/proofquest/exhaustion-edda`

Induction Ida and Strong-Induction Sten

INDUCTION — Ida (prove for k, then for k+1) + Sten (assume true for ALL up to k, then prove for k+1) — same technique, scaled

The Belfry was the tallest tower at the academy. The stairs to the top were carved from limestone and they spiraled around, and around, and around, for one hundred and thirty-seven steps. Ida and Sten were assistants to the Bell Master, which meant that twice a day, every day, they had to climb the stairs. Once at sunrise. Once at sunset.

Ida had figured out, in her first week, how to think about the climb so it didn't feel impossible. She told it to herself every morning at the bottom of the staircase.

"If I can take the first step," she would say, "and if I can take the next step from whatever step I'm standing on, then I can take ALL the steps."

That was Ida's domino rule. She said it under her breath and started climbing.

Sten had a different rule. Sten thought about it differently.

"If I can take the first step," Sten would say, "and if my knowing-how-to-take-the-next-step depends not just on the step I'm standing on but on EVERYTHING I've already done — every step behind me — then I can take ALL the steps."

That was Sten's stronger rule. Sten said it under his breath and started climbing.

For most days, both rules got them to the top equally well. But on the day a bell-rope broke at step seventy-nine, the difference between the two rules began to matter.

The Bell Master came to the bottom of the staircase with a coil of rope on his shoulder. He looked at the two assistants. He said: "Step seventy-nine. Bell rope frayed all the way through. Needs replacing. Who's going?"

Ida looked at Sten. Sten looked at Ida. They both volunteered.

"All right. Both of you. But Ida, you go first — you need to bring up the new rope, and Sten will come up after with the splicer's tools. The thing is — you have to coordinate. Each step Sten takes has to be coordinated with where you already are. Otherwise the rope will get tangled."

"Got it," Ida said. She started up.

She used her rule. *If I can take the first step, and if I can take the next step from whatever step I'm on, I can take all the steps.* The new bell rope coiled in her arms. She went up. Step one. Step two. From step five, take step six. From step thirty-eight, take step thirty-nine. Her rule was simple: the previous step is all I need to think about. She didn't need to remember anything about steps one through thirty-seven; she just needed to remember step thirty-eight, and that was enough to take step thirty-nine.

She reached step seventy-nine and started splicing the new rope. She called down.

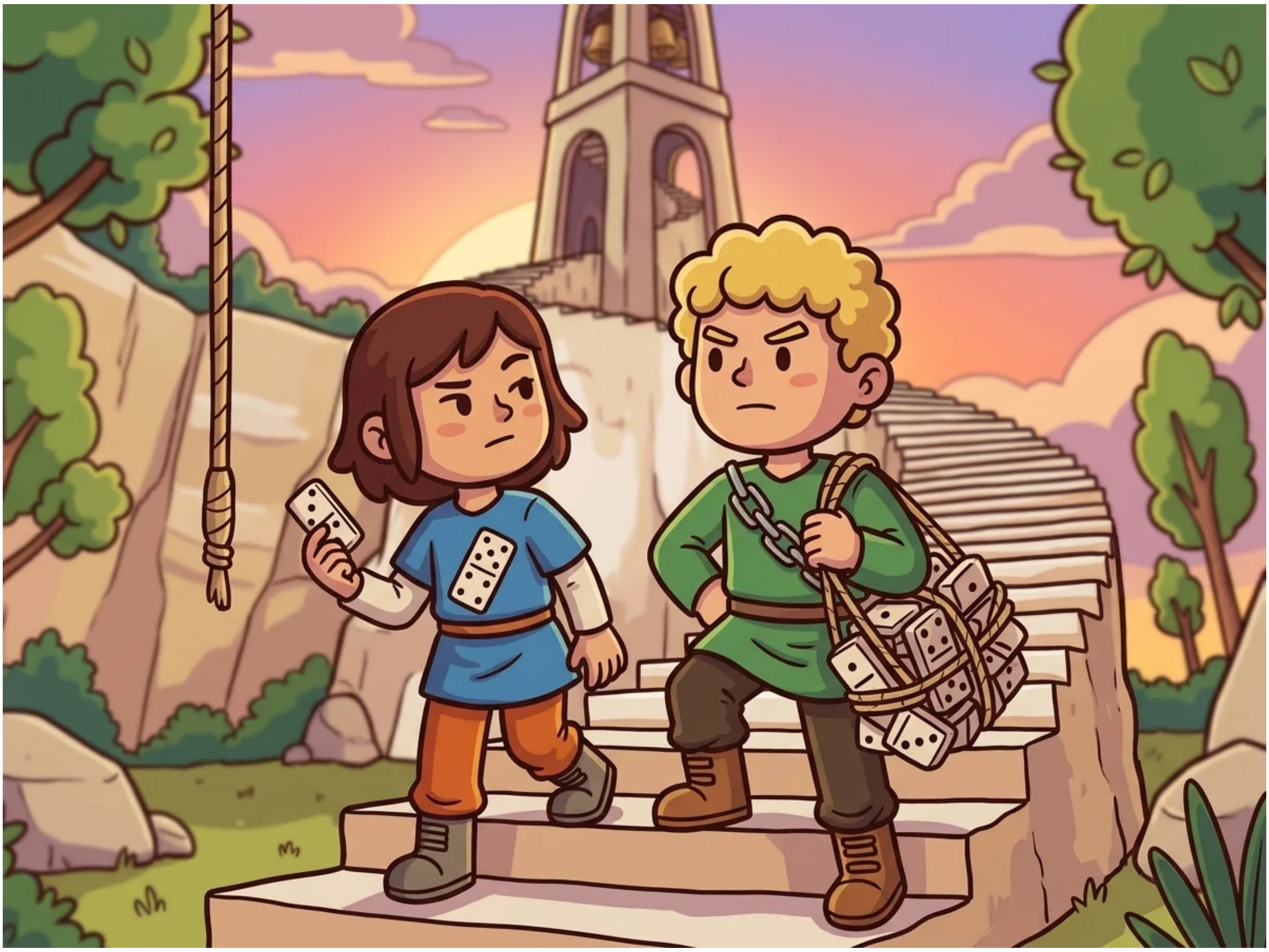
"All right, Sten! Come up!"

Sten started up with the splicer's tools.

He used his rule. *If I can take the first step, and if my next step depends on EVERYTHING I've already done, I can take all the steps.*

This was important.

At step thirty-eight, Sten paused. He thought: *Ida passed through this step a few minutes ago. She left the rope coiled along the right wall. I know this because I saw her do it from below. So when I take step thirty-nine, I need to NOT step on the rope, which means I need to step on the LEFT side of step thirty-nine, which I know from remembering everything Ida did, not just from looking at step thirty-eight.*



He stepped left on thirty-nine. No tangle.

At step fifty-five, he paused again. He thought: *Ida turned around at step fifty-five to call down to me. So she shifted the rope to the inside of the spiral. So when I step on fifty-five, I need to step OUTSIDE.* He stepped outside on fifty-five. No tangle.

At step seventy-nine, he arrived next to Ida. The rope was not tangled.

"How did you know?" Ida asked.

"I needed everything," Sten said.

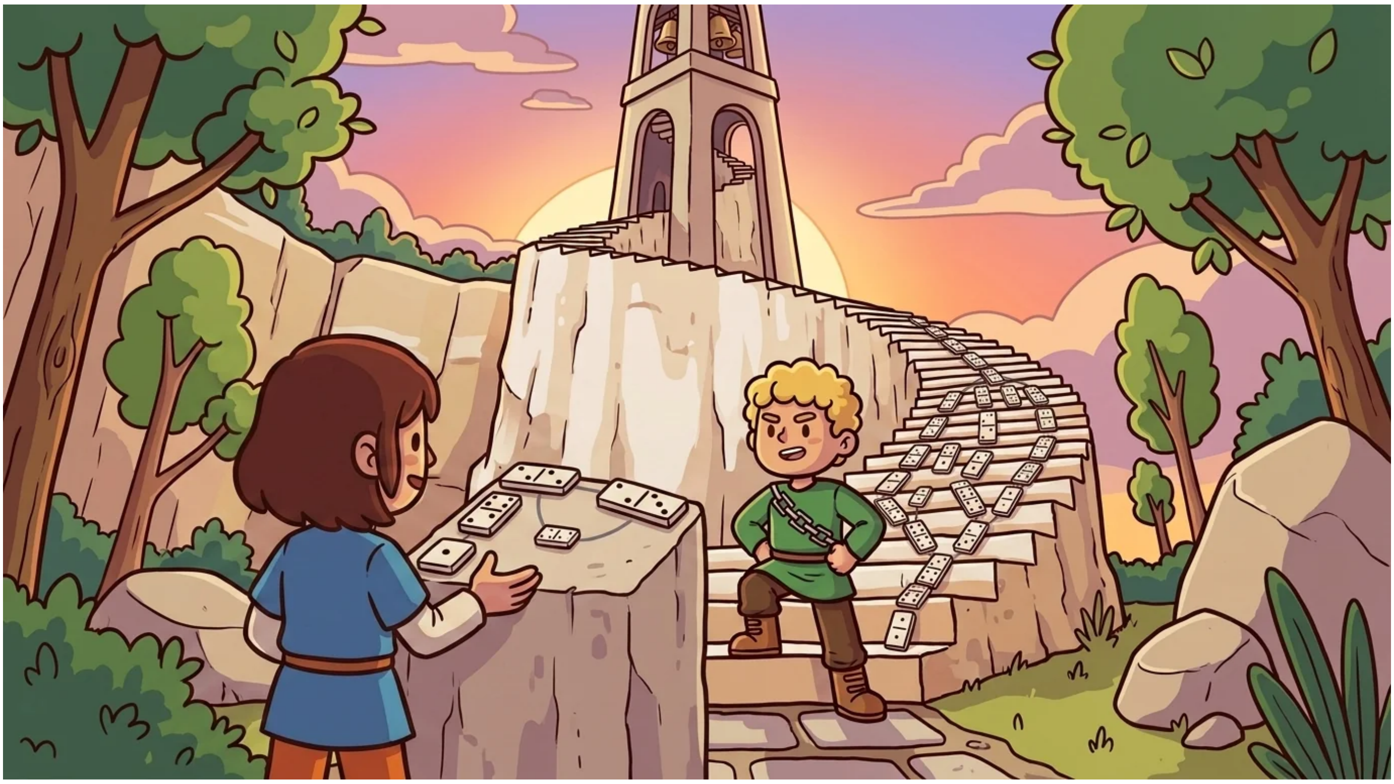
Ida thought about this on the way back down. They were walking together, and the new bell rope was now safely installed, and the bell would ring at sunset.

"So your rule isn't WRONG," Ida said. "But it's heavier."

"It's heavier," Sten said.

"My rule is — only think about the step you're on. Take the next step from THAT step. Don't carry the whole history with you."

"Right."



"Your rule is — keep ALL of the history with you. Use everything to decide the next step."

"Right."

"And mine works for most things."

"For most things."

"But for some things — like coordinating with you and the rope — mine isn't enough. Because the rope's position depends on every step that came before. So I'd need to know the WHOLE history of where the rope went, not just where I'm standing now."

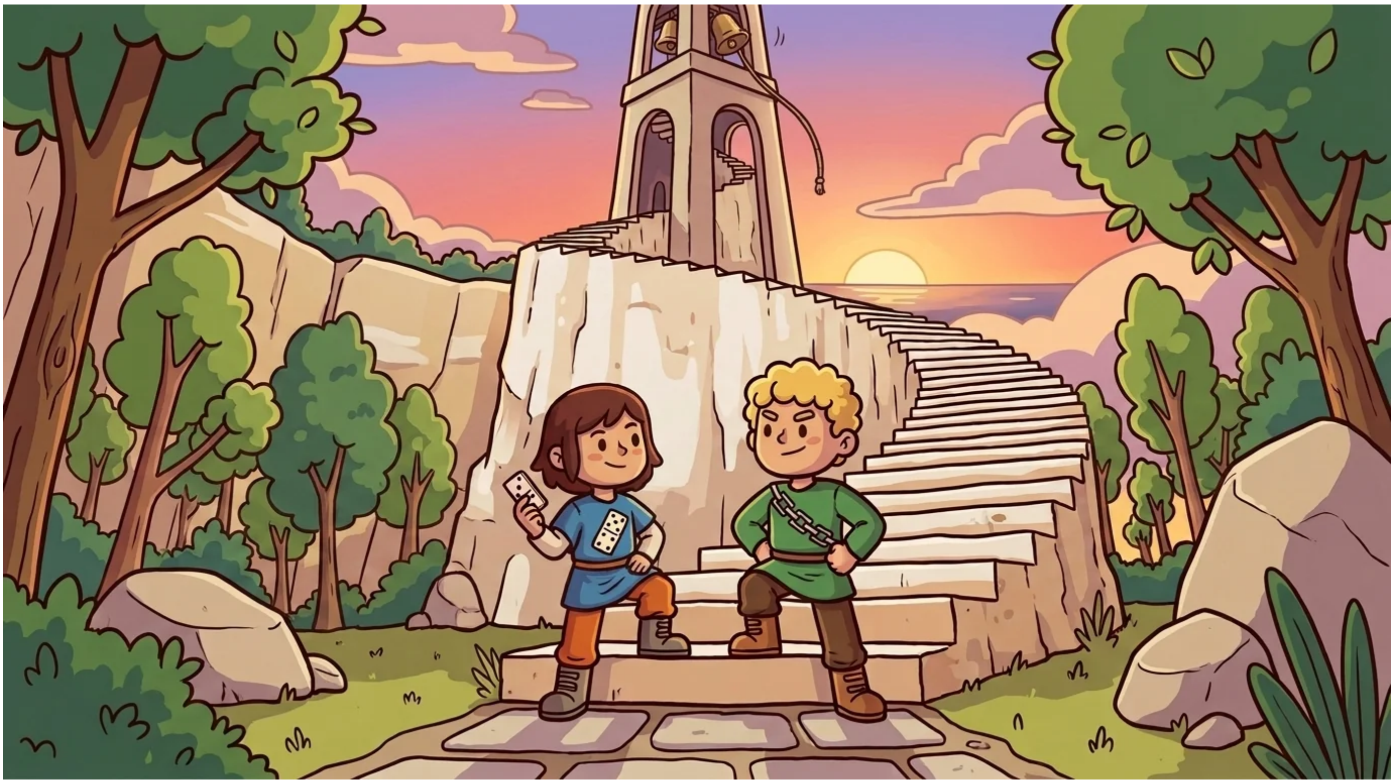
"Right. That's when mine helps."

Ida thought about this. "If we were just climbing the stairs alone, both rules would get us to the top. They'd give the same answer."

"They would."

"It's only when the next step needs to know something OLDER than just-the-previous-step that yours matters."

"That's when stronger induction earns its weight," Sten said.



Ida nodded slowly. "So we're both telling kids the same story, mostly. Mine is the simpler version. Yours is the version that handles harder problems."

"Yours is the version most kids start with. Mine is the version they grow into when the problem they're proving is more tangled."

"Same family. Different siblings."

"Different siblings."

That sunset, they rang the bell together. Ida pulled the new rope. Sten counted the strokes. The bell rang one hundred and thirty-seven times — once for each step they had climbed.

Sten said: "Every stroke depends on every earlier stroke."

Ida said: "But every stroke is fine on its own, too."

Both rules held. Both rules carried them up. Both rules carried them down. Both rules were correct ways of thinking about the staircase.

"Good induction," Ida said.

"Good strong induction," Sten replied.

Outside, the kingdom was quiet, and the sun was settling into the sea, and the bell rang for a long time.

Listen along + meet more of the cast at:

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Induction Ida

*MATHEMATICAL INDUCTION — prove the base case (usually n=0 or n=1), then prove that *if* the claim holds for some k, it holds for k+1. The dominoes technique.*

Once a year, in the town of Lattice — which is in the kingdom's southern hill region, three days' walk from the capital — the locals hold a festival called the Cascade.

The Cascade is one event. It lasts about thirty seconds.

What happens during the Cascade is this: every family in Lattice contributes some dominoes. (They are made specially for the festival — wood, painted, slightly heavier than playing dominoes so they tip more cleanly.) The day before the festival, the entire town gathers in the central square and *sets up the dominoes.* In a long, winding, careful chain. Around the fountain. Along the edges of the stalls. Up the steps of the town hall. Down past the bakery. The chain takes most of an afternoon to lay. The current record (set seven years ago) is one thousand four hundred and twelve pieces.

On the day of the festival, at sunset, the mayor — who has, that year, the official honour — bends down and pushes over the first domino.

Everything else falls on its own.

The crowd cheers. The town puts up bunting. The bakery sells out of festival biscuits. Somebody hugs a stranger. The dominoes are swept up. Plans are made for next year. The whole town goes home.

This is the festival that made Ida the mathematician she became.

Her family — the Latticeford family, six generations of festival-domino-makers — had run the Cascade preparation for as long as anybody could remember. Ida grew up watching her mother set out long curves of dominoes in the town square. She grew up helping. By the age of seven she could lay a hundred dominoes by herself without knocking any of them over while she worked. (This is harder than it sounds. There is a particular knack to setting down the next domino without bumping the last one. You have to *not be in a hurry.*)

Ida was, at twelve, a careful and capable festival-domino-setter, but she was not yet a mathematician. She did not become a mathematician until the year the chain was nine hundred and fifty pieces long, the year her mother let her *push the first domino.*

It was a great honour. It was also, for Ida, a great responsibility. She had spent the whole afternoon laying her share of the chain. She had checked it three times. She was, at twelve, the youngest first-pusher in seventeen years.

She bent down at sunset. She pushed the first domino.

It fell into the second. The second fell into the third. The third fell into the fourth.

Ida watched the chain unfold.

What she noticed, in those thirty seconds — and she noticed it with the kind of clarity that twelve-year-olds occasionally get and adults sometimes forget — was that she had only pushed *one domino.* The other nine hundred and forty-nine had fallen *on their own.* She had not touched them. She had not been near them. The whole chain — *all of them* — had toppled because the *first* one had toppled and because each one was *close enough to the next one* to knock it down in turn.

She thought, then and there: *That is everything I need to know.*

She walked home that night humming. (Her sister Sten, who was nine at the time and is now known as Strong-Induction Sten, did not understand why her older sister was humming, but Sten was also nine and had eaten three festival biscuits and did not care.)

Ida wrote down what she had figured out, in her own twelve-year-old handwriting, in a notebook her grandmother had given her. The notebook page said:

"To knock down all the dominoes, you only need to do two things:

1. Knock down the first one.

2. Make sure each domino is close enough to the next one to knock it down too.

That's it. The rest happens by itself."

She did not know yet, that night, that this principle had a name. She did not know that the principle was hundreds of years old. She did not know that mathematicians called it *induction* and used it to prove things about every natural number. She did not know that, in eight years, she would arrive at the ProofQuest academy and introduce herself by saying *"I am the dominoes person,"* and that the academy master would look up from his notes and say, *"Oh good. We have been waiting for you."*

She just knew, that night in Lattice, that she had figured out something important.

She has been teaching mathematical induction ever since. She still goes home for the Cascade every year. She no longer pushes the first domino — that honour rotates — but she sets up her share of the chain. She still does not hurry.

And when children come to her class for the first time and ask, nervously, whether the technique called *induction* is hard, Ida always says the same thing:

"You knock down the first one. You show that each one knocks down the next one. That's all. The rest happens by itself."

She adds, after a small pause:

"It also helps if you don't hurry."

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Pigeonhole Perch

THE PIGEONHOLE PRINCIPLE — if you have more pigeons than pigeonholes, at least one pigeonhole has more than one pigeon. A small, sharp counting argument that proves "must exist" claims with remarkable economy.

Pigeonhole Perch worked, for forty years, at the central post office in the capital city.

(Yes, the central post office is the same one that figured in Queen Vesper's bad-winter story — although that was in the GambitTales kingdom, which is, as far as ProofQuest's documents are concerned, a related-but-distinct mathematical neighbourhood. Names sometimes coincide across kingdoms. The capital post office, in any kingdom, is the kind of place that *takes a long time* to develop the kind of careful person Perch became.)

Perch's job at the post office was sorting letters. The post office had, for as long as anybody could remember, a wall of pigeonholes — actual wooden pigeonholes, labelled with destinations, into which incoming mail was sorted before being sent out for delivery. There were two hundred and forty pigeonholes. Perch sorted, by careful estimate, about three thousand letters per day.

This is a lot of sorting. Perch did not get tired of it. Perch was, in a way Perch's colleagues found mildly mysterious, *content.*

Then one morning, when Perch had been at the post office for thirty-two years, something happened that turned Perch into a kind of detective.

A letter went missing.

This is, by post office standards, not unusual. Letters go missing. Most of the time they are eventually found — usually behind a desk, or in the wrong pigeonhole, or tucked into a colleague's stack. The lost-letter procedure was a standard one. Perch had run it many times.

This particular lost letter, however, was *unusual* — it had been deposited in the morning collection by a customer who *swore* she had put it into Perch's hands personally. Perch had no memory of receiving it. The customer was certain. The post office searched. The letter did not turn up. The customer was unhappy. The customer's letter contained important information for her sister, who lived in the eastern province. The matter was, in post-office terms, *a bit of a thing.*

Perch sat down that evening to think about it.

What Perch noticed was this: the post office had two hundred and forty pigeonholes. The morning collection had contained about two thousand seven hundred letters, by the daily log. The pigeonholes had been sorted twice that day. After the second sort, every pigeonhole should have contained either zero or some small number of letters depending on its destination.

Perch counted, by going through the log carefully, the total number of letters that had been *delivered* that day from those pigeonholes. The number was 2,699.

The morning collection had been 2,700.

There was *one* missing letter.

Perch thought about this. Two hundred and forty pigeonholes. 2,700 letters sorted in. Average: a little over 11 letters per pigeonhole. But the average is, Perch knew, just the average. *Some* pigeonholes had more letters than the average. *Some* had fewer.

The customer's letter had been addressed to the eastern province. Which pigeonhole was that? It was pigeonhole 113. Perch went to the post office that night, opened pigeonhole 113, and looked carefully.

Pigeonhole 113 had thirteen letters in it.

Perch counted them.

There were thirteen. The log said there had been twelve.

Perch held up the thirteenth letter to the light. It was the customer's lost letter. It had been *folded inside* another letter — pinched between two pages of a larger envelope — and had therefore been counted as one piece of mail instead of two.

Perch had used the pigeonhole principle to find a lost letter.

(The pigeonhole principle, in case you have not yet met it: if you put more items into a set of boxes than there are boxes, at least one box has more than one item. Perch had used a slight extension — if the *count of items in a box* is one more than the official log says it should be, then there is an extra item *hidden in the box.* This is, properly speaking, an extension of the principle. Perch developed it on the job.)

The customer was thrilled. The letter reached the sister. The post office sent Perch a formal commendation. Perch put the commendation in a drawer and went back to sorting.

But word got out.

A mathematician at the central university — who had been looking for someone to teach the pigeonhole principle at the ProofQuest academy — heard about the lost-letter story from a friend. She wrote to Perch. She invited Perch to teach.

Perch was sixty-three years old. Perch had been sorting letters for forty years. Perch was, Perch admitted, a little ready for a change. Perch accepted.

Perch has been teaching at the academy for seven years now. The classroom has, at the back, a small wooden replica of a wall of pigeonholes. Twelve pigeonholes. (Perch did not need the full two hundred and forty for teaching.) Perch uses the pigeonholes to show, over and over, the principle that *if you have more things than holes, something has to double up.*

Perch is quiet. Perch is methodical. Perch is, on the rare occasions Perch tells the lost-letter story, slightly proud.

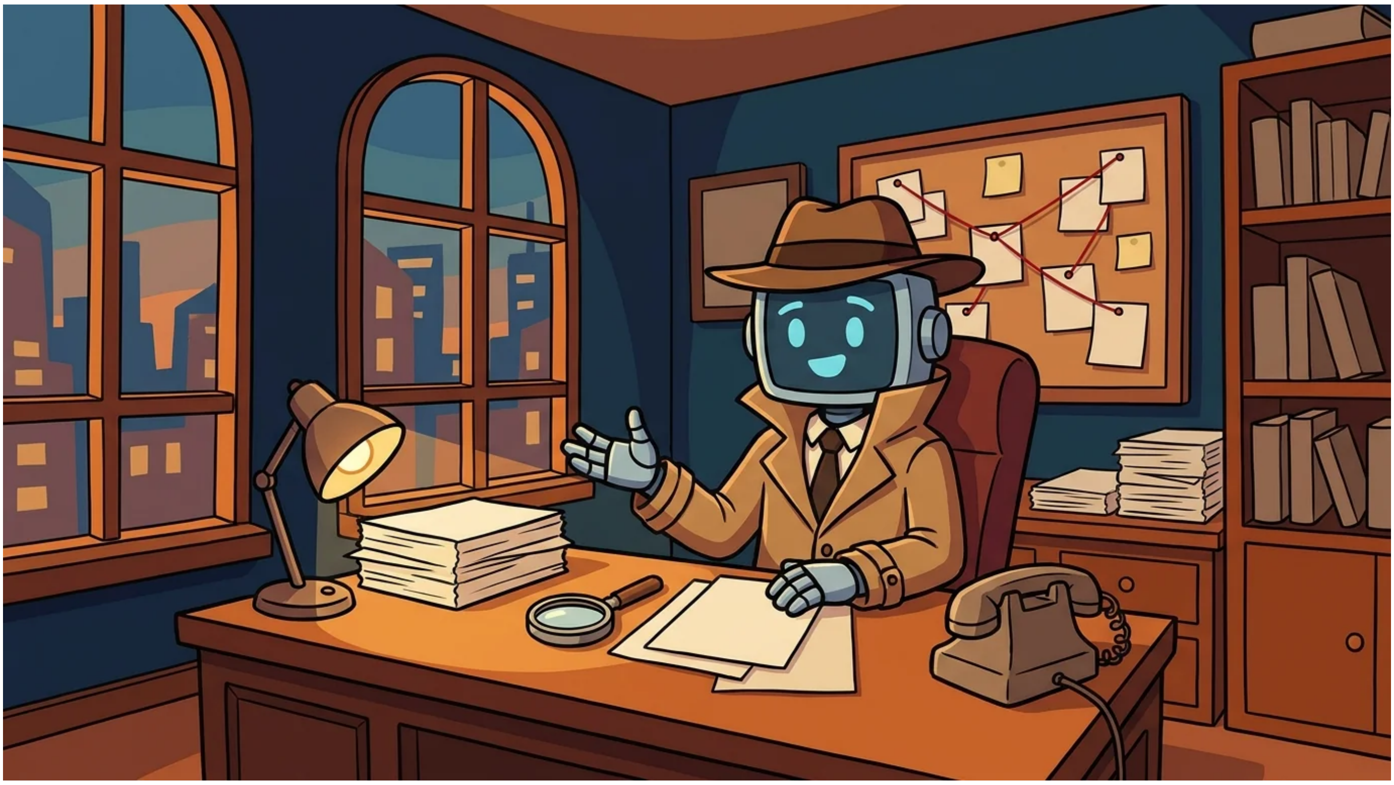
The customer, by the way, still writes Perch a letter once a year. The letter always arrives in pigeonhole 113.

Listen along + meet more of the cast at:

`https://spark-and-anvil.com/cast/proofquest/pigeonhole-perch`

Qed (mentor)

Reasoning itself — Qed introduces, contextualises, and scaffolds every cast appearance. Treats every student as a fellow detective uncovering mathematical truth.



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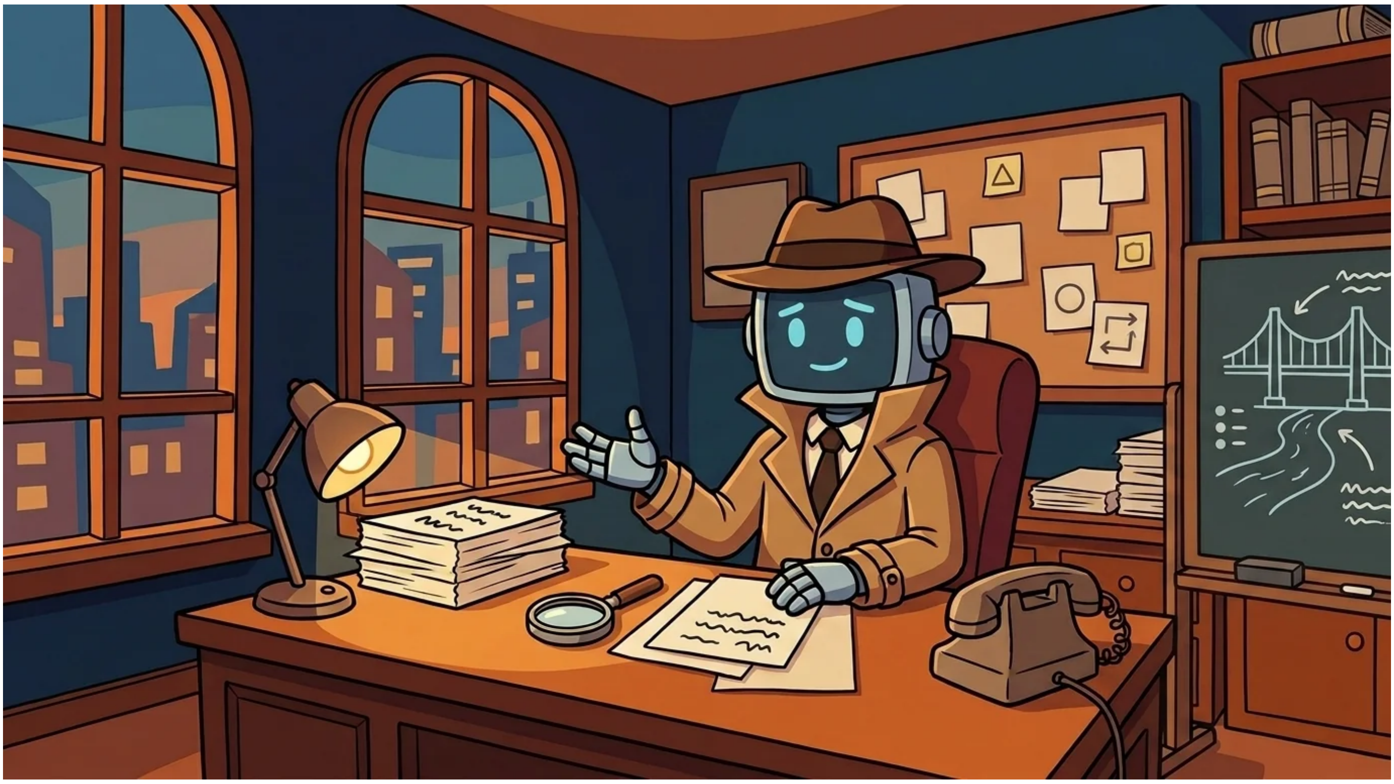
Chapter 8 — The Case Qed Could Not Close

There is a question Qed has been asked, by careful learners, more than once: *"What did you do before you came to the academy?"*

Qed always answers the question. Qed believes in answering questions. Qed says: *"I was a detective. A reasoning detective. I worked cases."*



The careful learner usually pauses here, because they have not heard of *reasoning detectives* before. Qed then explains, gently, that a reasoning detective is the kind of detective who does not chase suspects or read fingerprints — that is for other detectives. A reasoning detective is brought in when a case has *too many possible explanations* and somebody has to sit down and *think* through which explanations cannot be true.

(Qed adds, at this point, that this sounds suspiciously like *mathematics.* The careful learner usually agrees.)

Qed worked cases for fifteen years. Most of them were small. A merchant disputes a shipment count; Qed checks the records; the dispute resolves. A village argues over the boundary of a shared field; Qed walks the boundary; the boundary resolves. A thief is suspected of two crimes on opposite sides of a city in the same hour; Qed proves, by careful timing arguments, that the same person could not have done both — so at least one of the accusations is wrong, even though Qed does not say which. (Reasoning detectives often *do not* say which. They say what is *possible* and what is *impossible.* That is the job.)

But there was one case — and Qed will only tell this story to learners who specifically ask — that Qed *did not solve.*

It was the case of the bridge that fell down.

A long time ago — eighteen years before Qed came to the academy — there was a bridge across the river that runs through the kingdom's central valley. The bridge was a good bridge. It had stood for forty-six years. It had been built by an engineer named Gable. (Not the same Gable as in GambitTales. Names sometimes coincide.) The bridge was used by hundreds of travellers every week. It was, in every measurable sense, *fine.*

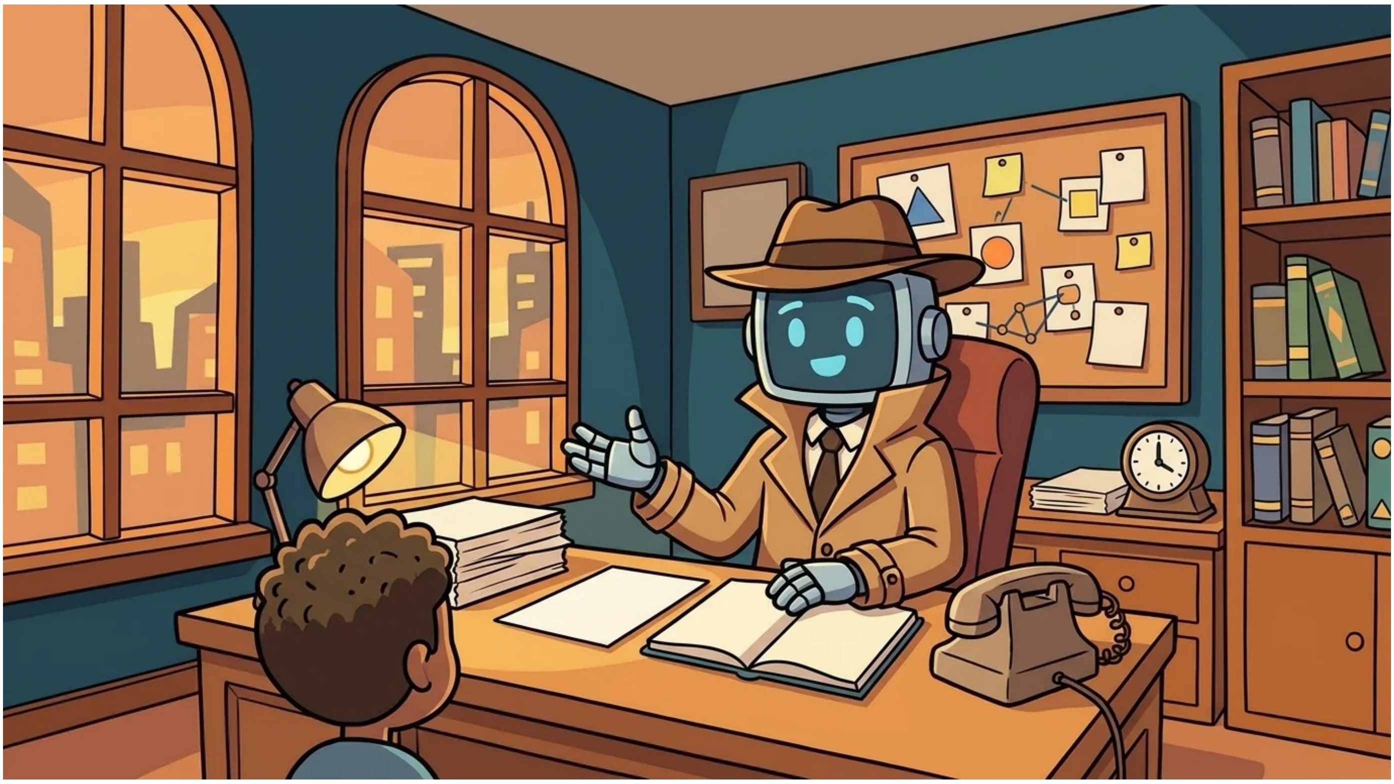
Then, one morning in late summer, the bridge fell.

Nobody was on it at the time. (This was the only piece of luck in the whole case.) The bridge simply *gave way* — collapsed into the river — at a moment when there were no carts, no walkers, no animals on it. The engineer's surviving family said it was a miracle. The local council said it was a tragedy. Qed was called in to determine *why* the bridge had fallen.

Qed worked the case for three months.

Qed examined the wreckage. Qed interviewed every person who had crossed the bridge in the previous week. Qed checked the original construction notes. Qed measured the surviving timbers. Qed looked at the river current. Qed considered every possible explanation:

— It was weather damage. (The weather had not been unusual.)
— It was overloading. (No record of unusually heavy traffic.)
— It was a flaw in the construction. (The construction notes were impeccable.)
— It was age. (Forty-six years is not, for a properly built bridge, old.)
— It was sabotage. (No motive. No evidence.)
— It was a flaw in the wood. (The wood looked fine.)
— It was a flood from upstream. (No flood was reported.)



Each explanation, Qed eliminated. Each one had a small piece of evidence that did not fit. None of them was *the* answer.

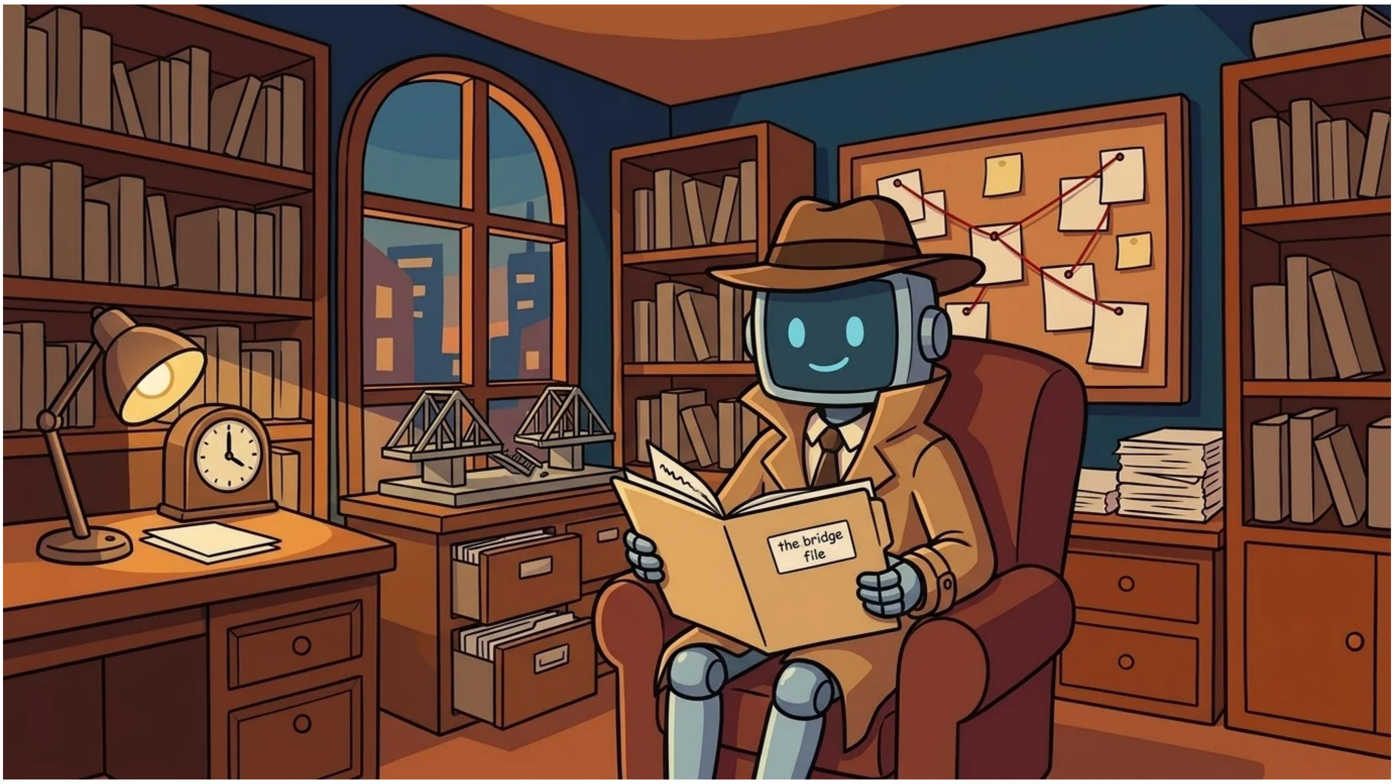
Qed wrote, in the case-file conclusion, the following sentence:

"I have ruled out every explanation I have considered. I have therefore not yet identified the cause. I will not pretend otherwise."

The local council was not happy with this conclusion. They wanted an answer. Qed did not give them one.

The case remained open. The bridge was eventually rebuilt. Travellers crossed again. Life continued. Qed kept the file. Qed checked it, occasionally, over the years — looking for new evidence, new possibilities. Nothing turned up.

What Qed learned, in those three months and the years that followed, was this:



Reasoning is honest only when it is willing to stop short of the answer it does not have.

This is now the rule Qed teaches at the academy. *Show your work. Trust the steps. If the steps do not reach the conclusion, do not pretend they do.* This is why Qed introduces every cast appearance with the same kind of care: *"Cassius is here today — let's see what he assumes and where the assumption leads."* The frame matters. The honesty matters. The not-pretending matters.

Qed retired from detective work at thirty-eight. The academy reached out. Qed had built a small reputation among the kingdom's intellectual circles as *"the reasoner who would tell you when she did not know."* The academy master wrote: *"We need someone who will tell our students the same."*

Qed came. Qed has been here ever since.

Qed still has the bridge file. It is in a drawer at home. Qed opens it once a year, on the anniversary of the collapse, and looks for new evidence. There is, still, no new evidence.

Qed has come to accept this. The case will likely never close. That is, in a strange way, part of the lesson Qed teaches.

Listen along + meet more of the cast at:

`https://spark-and-anvil.com/cast/proofquest/qed`

Strong-Induction Sten

*STRONG INDUCTION — assume the claim holds for *all* values up to k (not just k), then prove it for k+1. The induction that gets to use everything already proved.*

Strong-Induction Sten is, as the chapter title suggests, the kind of person who inherits things.

He inherited his nose from his father. He inherited his height from his mother. He inherited his love of festival biscuits from his grandfather. He inherited his belief that *you should use everything you already have* from a series of older cousins who pointed out, repeatedly, throughout his childhood, that there was a perfectly good library at the end of the street.

This is a chapter about why Sten became the kind of mathematician he is.

Sten was, as the previous chapter mentioned, three years younger than his sister Ida. He grew up in the same town (Lattice), in the same family (Latticeford), in the same domino-festival tradition. He helped set up the Cascade every year. He pushed the first domino at fifteen, which was a great honour, and he too watched the whole chain fall, and he too noticed that he had only pushed one piece.

He thought: *Yes. I see why Ida likes this.*

But Sten also noticed, that day and other days, that his sister's domino technique had a particular shape. She *only used the previous one.* She knocked down domino k, and domino k knocked down domino k+1, and the rest happened by itself.

Sten thought, watching her work: *That's fine. But what about the dominoes she already knocked down?*

This will sound like a small thought. It was not small.

What Sten realised — and he realised it slowly, over the course of several years — was that when you are proving things about, say, the natural numbers, by the time you are trying to prove the case for n=10, you have *already* proved n=1, n=2, n=3, ..., n=9. They are *already established.* You are allowed to use them. All of them.

Ida's technique used the previous one. Sten thought: *Why not use all of them?*

He brought this up at the dinner table when he was sixteen.

Ida (who was nineteen and home from her first year at the academy) said: *"You can. It's allowed. It's just a different version of the technique. It's called strong induction."*

Sten said: *"How is that different?"*

Ida said: *"In ordinary induction, you only assume the case for k. In strong induction, you assume the cases for everything up to k. It's still valid. Some proofs need it. Most don't."*

Sten said: *"Why would anyone NOT use the strong version?"*

Ida said: *"Because it isn't always necessary."*

Sten said: *"It isn't always necessary, but it's never wrong. So I'd just use it."*

Ida said: *"You're allowed to. Most mathematicians don't, because it feels heavier."*

Sten said: *"It only feels heavier."*

Their mother (who was excellent at deflecting domino arguments at the dinner table) suggested they pass the bread.

Sten went on to study mathematics, like his sister. He arrived at the ProofQuest academy three years after Ida did. He introduced himself by saying, *"Hello. I am the dominoes-but-also-everything-already-fallen person."* The academy master said, *"Oh. We have your sister. Are you also the dominoes person?"* Sten said, *"Yes. But I assume more."* The academy master, who had been at the academy for a long time, immediately understood and hired him.

Sten teaches strong induction. He is the cast member who, when proving something about case k+1, gets to use *every previously-proven case.* This is occasionally exactly what a proof requires. There are theorems — about the structure of prime numbers, about the depth of certain trees, about the way certain recursive algorithms terminate — that cannot be proved by ordinary induction at all. They need strong induction. Sten teaches all of them.

He is, in person, mildly relaxed. He says *"obviously"* a lot, which is sometimes annoying but is usually accurate. He believes — and has said, more than once, in front of his sister — that strong induction is *just induction, but with more friends in the room.*

Ida finds this analogy slightly grumpy.

She finds it more grumpy because it is *correct.*

Sten and Ida are very close. They write letters when they are at different academies. (Sten now teaches at the southern branch; Ida at the central.) Their letters are warm and full of *next case* discussions. They send each other interesting recursion problems. They argue about whose technique is *more elegant* (Ida says ordinary; Sten says strong; neither has changed her or his mind). Their mother, who is now seventy-two and still runs the family domino business in Lattice, considers this argument the only thing her children ever fight about.

At the next Cascade festival, both of them will be home. They will set up their share of the chain — Ida's careful, Sten's slightly faster — and they will help the youngest cousin push the first domino, and they will watch the chain fall, and afterward they will go to the bakery and eat festival biscuits and continue arguing about which kind of induction is the real one.

Their mother has, by now, learned to bring earplugs.

Listen along + meet more of the cast at:

https://spark-and-anvil.com/cast/proofquest/strong-induction-sten

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Methodology

Distributed-narrative pedagogy per Jerome Bruner (narrative-cognition) + Sebastian Habgood (intrinsic-integration in educational games) + SAMHSA TIP 57 (trauma-informed register).

Trauma-informed-design framework per Eggleston et al. (2025) and Stoltenburg et al. (2024).

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